entropy

$$Tds = (1/\rho^\circ)\,\sigma : \mathbf{A}_e^\circ \Phi \mathbf{A}_p^{\circ -1} \mathbf{A}_e^{\circ -1} (1 - V_p)^{-1/2} - (1/\rho^\circ)\, p V_p^{\cdot}$$

The first term on the right-hand side is positive by virtue of the demands imposed on Φ, and the second term by virtue of the reasons already listed. Thus the second law of thermodynamics retains its validity for the model of a viscoelastic fracturing medium. Relations (1), (2), (4) and (8) form a complete system of equations describing the motion of a viscoelastic medium with fracture.

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BOUNDS ON CONTROL IN THE LINEAR DYNAMIC OPTIMIZATION PROBLEM WITH A QUADRATIC FUNCTIONAL*

B.N. SOKOLOV

Some bounds of the region from whh a linear system can go in a prescribed time to the origin with a given value of an integral functional that is quadratic in control are derived. A bound on the required control is given. Conditions are proposed when a controller can be designed taking any point from a given bounded region to the origin in a prescribed time with control not exceeding the specified bound.

The design of programmed bounded controls that take a linear system in a finite time to a prescribed state is considered in /1, 2/.

Consider a linear controlled system

$$x^{\cdot} = Ax + bu \tag{1}$$

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Here x is an n-dimensional vector, A is a constant matrix, b is a vector, and u is the scalar control. System (1) is assumed to be completely controllable, i.e., the vectors b, Ab, $\ldots$, $A^{n-1}b$ are linearly independent /3/.

Let $x(0) = x_0$ be the location of the controlled object at time $t_0 = 0$ and $u(x_0, T-t)$ the control that takes the phase vector from x_0 to the origin in time T and such that

$$\mu_0^2 - \mu^2(0) \geqslant 0, \ \mu_0^2 = \text{const}$$
$$\mu^2(t) = \int_t^T u^2(x(t), T-\tau)\,d\tau \tag{2}$$

Denote by $D(\mu_0^2, T)$ the region from which system (1) may reach the origin in time T with a value of integral (2) not exceeding μ_0^2. We know /3/ that $D(\mu_0^2, T)$ is an ellipsoid centred at the origin. Let us derive a lower bound on the radius of the region $D(\mu_0^2, T)$, i.e., find the radius of the sphere which is completely inscribed in this region. Reversing the time in Eqs.(1), we consider the region $D'(\mu_0^2, T)$ reachable from the origin by the system

$$x^{\cdot} = -Ax + bu, \ x(0) = 0 \tag{3}$$

in time T with value of the integral (2) not exceeding μ_0^2. The region D' clearly coincides with D.

The optimal programmed control $u(0, T-t)$ that takes system (3) in time T from the origin to the boundary of the reachability region $D(\mu_0^2, T)$ may be represented in the form /3/

$$u(0, T-t) = U(T-t), \ U(T-t) = l^* \exp(-A(T-t))\,b$$

Here $\exp(-A(T-t))\,b$ is the impulse transition function of system (3), and the zero inside the parentheses in u indicates that the motion starts at the origin. The vector l is defined by the right-end boundary conditions and the constraint $\mu_0^2 - \mu^2(0) \geqslant 0$ on the value of the integral (2) for $t = 0$ and $x(0) = 0$. We finally obtain the represention

$$u(0, T-t) = \mu_0 U(T-t)\, I^{-1/2}, \quad I = \int_0^T U^2(T-\tau)\,d\tau \tag{4}$$

Substituting the control (4) into the equation of motion (3), we use the Cauchy formula to find the maximum displacement of the phase point from the origin in the direction l:

$$l^* x(T) = \int_0^T U(T-\tau)\, u(0, T-\tau)\,d\tau = \mu_0 I^{1/2}$$

For the minimum radius R of the reachability region $D(\mu_0^2, T)$ we thus obtain the equation

$$\mu_0^{-2} R^2 = \min_l I, \ |l| = 1 \tag{5}$$

We will use the following notation: $t_0 = 0$, $t_{i+1} = t_i + 1$, $i = 0, 1, \ldots, N$, and $N = [T]$ is the integer part of T. Let us transform and estimate the expression on the right-hand side of (5). It is not less than

$$\min_l \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} U^2(T-\tau)\,d\tau = \min_l \sum_{i=0}^{N-1} \int_0^1 (l^* \exp(-A(T-t_i)) \exp(A\tau')\, b)^2\, d\tau'$$

In the last integral, we have made the change of variables $\tau_i = t_i + \tau'$. The corresponding sum of the integrals is given by

$$\min_l \sum_{i=0}^{N-1} \int_0^{-1} (\zeta_i^* \exp(A\tau)\, b)^2\, d\tau\, (l^* \exp(-A(T-t_i)))^2 \geqslant c_0 \sum_{i=0}^{N-1} \min_l (l^* \exp(-A(T-t_i)))^2 \tag{6}$$

$$\zeta_i^*(l) = l^* \exp(-A(T-t_i)) / |l^* \exp(-A(T-t_i))|, \quad i = 0, 1, \ldots$$

$$c_0 = \min_l \int_0^1 (l^* \exp(A\tau)\, b)^2\, d\tau, \quad |l| = 1$$

and $c_0 > 0$, since system (1) is completely controllable /3/. The second factor in the last expression in (6) is

$$\sum_{i=0}^{N-1} [\max_l (l^* \exp(A(T-t_i)))^2]^{-1} = \sum_{i=0}^{N-1} \| \exp(A(T-t_i)) \|^{-2} \tag{7}$$

Assume that the characteristic values of the matrix A with the maximum real parts α have prime elementary divisors (assumption A). Then for $T - t_i \geqslant 0$ we have the bound /4/

$$\| \exp (A (T - t_i)) \| \leqslant c_1 \exp (\alpha (T - t_i)) \tag{8}$$

where the constant c_1 depends only on the matrix A. Using the bound (8) and inequalities (6) and (7), we finally obtain

$$R^2 \geqslant R_0^2 (\mu_0, T) = \mu_0^2 c_0 c_1^{-2} \sum_{i=0}^{N-1} \exp (-2\alpha (T - t_i)) \tag{9}$$

Lemma 1. If the matrix A satisfies assumption A, then the minimum radius R of the reachability region $D(\mu_0^2, T)$ of a completely controllable dynamic system (3) has a lower bound (9). If the homogeneous system (1) is stable, then R_0 increases without limit as T increases for fixed μ_0, and

$$R^2 \geqslant R_0^2 (\mu_0, T) = \mu_0^2 c_0 c_1^{-2} [T] \tag{10}$$

Proof. The first proposition follows from the previous bounds. From the stability of system (1) it follows that /4/ either $\alpha = 0$ and assumption A holds or $\alpha < 0$. In any event, the bound (8) holds with $\alpha = 0$. For $\alpha = 0$, relationship (9) reduces to the bound (10), which determines the corresponding increase in R_0.

Let us now obtain a bound on the programmed control $u(0, T - t)$ that takes the phase point of sytem (3) from the origin to the boundary of the region $D(\mu_0^2, T)$. From expression (4) for $T = 1, 2, \ldots$ we have the bound

$$| u(0, T - t) | / \mu_0 \leqslant \Big[\min_t \min_\zeta (\zeta^* \exp (At) b)^{-2} \sum_{i=0}^{N-1} \int_0^1 (\zeta^* \exp (A (t_i + \tau)) b)^2 d\tau \Big]^{-1/2}$$

$$\zeta^* = l^* \exp (-AT) / | l^* \exp (-AT) |$$

Here we have made the change of variables $t_i + \tau' = \tau$ in the integrals and then dropped the prime.

The right-hand side of the last inequality does not exceed

$$c_0^{-1/2} \Big[\min_t \sum_{i=0}^{N-1} [\max_{\zeta_i} (\zeta_i^* \exp (A (t - t_i)) b)^2]^{-1} \Big]^{-1/2} =$$

$$c_0^{-1/2} \max_t \Big\{ \sum_{i=0}^{N-1} \| \exp (A (t - t_i)) b \|^{-2} \Big\}^{-1/2}$$

$$\zeta_i (\zeta) = \zeta^* \exp (A t_i) / | \zeta^* \exp (A t_i) |$$

The expression in braces on the right-hand side of the last relationship for fixed t increases as $N = [T]$ increases. The sum is therefore a minimum when it consists of a single term, i.e., when $N = 1$. Using the bound (8), we thus obtain for $t \in [0, T]$

$$| u(0, T - t) | \mu_0^{-1} \leqslant c_0^{-1/2} \max_\tau \| \exp (A\tau) b \| \leqslant c_0^{-1/2} c_1 | b | \max_\tau \exp (\alpha \tau), \quad \tau \in [0, 1] \tag{11}$$

If system (1) is stable, i.e., $\alpha \leqslant 0$, then the maximum on the right-hand side of the bound (11) is attained for $\tau = 0$. We finally have

$$| u(0, T - t) | \leqslant u_0 = \mu_0 c_0^{-1/2} c_1 | b |, \quad t \in [0, T] \tag{12}$$

Lemma 2. Suppose we are given a family (4) of ccontrols $u(0, T - t)$ that takes the completely controllable system (3) from the origin to the boundary of the reachability region $D(\mu_0^2, T)$ in time $T \geqslant 1$. Then there exists a constant c_2, independent of T and μ_0, such that for all controls of the family (4) for all $t \in [0, T]$ we uniformly have $| u(0, T - t) | \leqslant u_0 = \mu_0 c_2$.

The proof follows from the bound (11) or the improved bound (12) if system (1) is stable. In the latter case, $c_2 = c_1 c_0^{-1/2} | b |$.

We know /3/ that synthesis in the linear problem with a quadratic functional realizes a linear controller with time-dependent feedback coefficients. On optimal trajectories, the values of the feedback control are indentical with the programmed control. This leads to the following theorem.

Theorem. Assume that the dynamic system (1) is completely controllable and the corresponding homogeneous system is stable. Then for any constants u_0 and R_0 specified in advance, we can select a termination time T such that the optimal linear controller corresponding to the minimum of the integral (2) takes the system (1) from any phase state $x_0 : | x_0 | \leqslant R_0$, to the origin with a control not exceeding $u_0 : | u(x(t), T - t) | \leqslant u_0$.

The proof follows from Lemmas 1 and 2. Given the constraint u_0, we obtain from inequality (12) the value $\mu_0 = u_0 c_0^{1/2} C_1^{-1} | b |^{-1}$, which guarantees the required control bound for any $t \in [0, T]$, $T \geqslant 1$. Then, using the bound (10), we compute R_0 and the time T that ensures the condition $| x_0 | \leqslant R_0 (\mu_0, T)$. To this end, it suffices to take $T \geqslant | x_0 |^2 \cdot \mu_0^{-2} c_0^{-1} c_1^2 + 1$. Substituting the corresponding value of μ_0, we finally obtain $T \geqslant | x_0 |^2 u_0^{-2} c_0^{-2} c_1^4 | b |^2 + 1$.

Example. Consider a point that moves with a bounded velocity along the horizontal directrix. Assume that the velocity of the point may change instantaneously within given bounds. There are m pendulums of various lengths attached to the point. Controlling the velocity of the point, it is required to move the system to the origin so that all the oscillations are damped. It can be verified that this system is stable and completely controllable (the controllability is proved in /5/). Therefore, for any bounded region in phase space, we can construct by our theorem a linear velocity controller which takes the system from any initial position in this region to the origin. The control on any of the realized trajectories will not exceed the specified value.

Remarks. 1°. If system (1) is unstable, then a control bounded by a given constant that takes a phase point to the origin does not necessarily exist. As an example, consider the equation $x^{\cdot} = x + u$, $| u | \leqslant u_0$, $x \in R^1$. For $| x_0 | > u_0$, the required control obviously does not exist. The conditions of the theorem are therefore very close to necessary.

2°. All the bounds and conclusions remain valid in the case when u is a vector and b is an appropriately dimensioned matrix.

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A SPECIAL CASE OF HYDRODYNAMIC STABILITY*

A.G. BERSHADSKII

The following dependence of the amplitude of the velocity perturbations on the supercriticality parameter: $A \sim \delta^{1/2}$ is typical of the case of the selfexcited oscillations which are generated when there is instability in the stationary flows of a viscous incompressible fluid. There is, however, a special case (it is investigated in this note) when this dependence is linear (as in the case of bifurcations in a stationary regime /1/). A condition is obtained for the existence of such selfexcited oscillations together with an algorithm which enables one to find their frequency and amplitude. In the case of these selfexcited oscillations there is a further difference from conventional hydrodynamic selfexcited oscillations in that sub- and supercritical regimes coexist in them and, at the same time, the subcritical selfexcited oscillations turn out to be unstable while the supercritical selfexcited oscillations are stable.

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